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# Weighted Uncertainty Principles in $L^\infty$

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We determine sufficient conditions on positive weights $W$ and $V$ such that there exists continuous, strictly increasing functions $\Phi$ and $\Psi$ on $[0, \infty)$ such that $\Phi(0) = 0 = \Psi(0)$ and

$$\Phi\left(\frac{\|Wf\|_\infty}{|\hat{f}(0)|}\right) \Psi \frac{\|V\hat{f}\|_\infty}{|f(0)|}\Bigg) \geqslant 1$$

whenever $f\colon R \to R$ is a continuous integrable function. We also give an example that shows the optimality of our conditions. 
*Key Words:* Poisson summation formula; uncertainty principle.

## 1. INTRODUCTION

The classical uncertainty principle roughly states that a function $f$ and its Fourier transform $\hat{f}$ cannot be both highly localized: there exists a constant $K$ such that for any real numbers $t_0$, $\xi_0$, and $f \in L^2$,

$$\|f\|_2^2 \leqslant K \,\|(t-t_0)\, f\|_2\, (\xi-\xi_0)\, \hat{f}\|_2.$$

Refinements have been obtained for various function spaces. For instance, Benedetto and Heinig obtained the following weighted uncertainty principle inequality: if $1 < p \leqslant q < \infty$ and positive even weights $u, v$ defined on $R$ satisfy

$$\sup_{s>0} \left(\int_0^{1/s} u(\xi)\, d\xi\right)^{1/q} \left(\int_0^s v(t)^{-p'/p}\, dt\right)^{1/p'} < \infty, \tag{1}$$



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then there exists a constant $C$, depending only on the supremum above such that

$$\|f\|_2^2 \leqslant C\left(\int_{-\infty}^{\infty} |tf(t)|^p\, v(t)\, dt\right)^{1/p} \left(\int_{-\infty}^{\infty} |\xi \hat{f}(\xi)|^{q'}\, u^{-q'/q}(\xi)\, d\xi\right)^{1/q'} \tag{2}$$

for any function $f$ from the Schwarz class. For example, see [1]. We also refer the reader to [2] for an excellent survey on the uncertainty principle.

The techniques used to prove inequalities like (2) involve integration by parts, Hölder's inequality, Plancherel's theorem and their generalizations. However, it is unclear how these can be used to obtain a reasonable version for $L^\infty$. The purpose of this note is to determine sufficient conditions on positive weights $W$ and $V$ so that there exist strictly increasing, continuous functions $\Phi$ and $\Psi$ on $[0, \infty)$ satisfying $\Phi(0) = 0 = \Psi(0)$ and

$$\Phi\left(\frac{\|Wf\|_\infty}{|\hat{f}(0)|}\right) \Psi \frac{\|V\hat{f}\|_\infty}{|f(0)|}\Bigg) \geqslant 1 \tag{3}$$

for any continuous integrable function $f: R \to R$.

The idea of the proof is simple: the Poisson summation formula is applied to obtain sign changes of the function

$$\theta_f(x) := \frac{2\pi}{|x|} \sum_{k \neq 0} f(2\pi k/x) - \int_{-\infty}^{\infty} f(u)\, du.$$

## 2. MAIN RESULTS

We define the Fourier transform of $f$ by

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt.$$

THEOREM 1. *Let $W$ and $V$ be even real-valued functions defined on $R$, nondecreasing and strictly positive on $(0, \infty)$ such that*

$$\int_0^1 \frac{dt}{W(t)} = \infty = \int_0^1 \frac{dt}{V(t)}, \tag{4}$$

$$\int_1^\infty \frac{dt}{W(t)} < \infty, \qquad \int_1^\infty \frac{dt}{V(t)} < \infty, \tag{5}$$

*and such that for some constants* $C_0$, $C_1$

$$W(2t) \leqslant C_0 W(t), \; V(2t) \leqslant C_1 V(t) \tag{6}$$

*for all real numbers t. Then there exist strictly increasing, continuous functions* $\Phi$ *and* $\Psi$ *defined on* $[0, \infty)$ *satisfying* $\Phi(0) = 0 = \Psi(0)$,

$$\lim_{x \to \infty} \Phi(x) = \infty = \lim_{x \to \infty} \Psi(x),$$

*and such that*

$$\Phi\left(\frac{\| Wf \|_\infty}{|\hat{f}(0)|}\right) \Psi\left(\frac{\| V\hat{f} \|_\infty}{|f(0)|}\right) \geqslant 1 \tag{7}$$

*is satisfied for all continuous integrable functions* $f\colon R \to R$.

Conditions (4) and (5) ensure that the functions

$$H(u) = \left(2C_0 \int_{2\pi u}^{\infty} \frac{ds}{W(s)}\right)^{-1} \qquad \text{and} \qquad G(u) = \left(\frac{C_1}{\pi} \int_u^{\infty} \frac{ds}{V(s)}\right)^{-1}, \qquad u > 0$$

are bijections of $(0, \infty)$ onto itself, a requirement for the construction of the functions $\Phi$ and $\Psi$. The proof of Theorem 1 shows that these functions may be taken as follows: $\Phi(t) = H^{-1}(A_\alpha t)$, $\Psi(t) = G^{-1}(B_\alpha t)$ where $A_\alpha = (\alpha C_0 + 1)(2\alpha - 1)^{-1}$, $B_\alpha = (2\alpha + 1)(1 - \alpha)^{-1}$ and $\alpha$ is any constant such that $1/2 < \alpha < 1$. Taking $W(x) = |x|^a$ and $V(\xi) = |\xi|^b$ with $a > 1$ and $b > 1$, we obtain the following corollary:

Corollary 1. *Given* $a > 1$ *and* $b > 1$, *there exists a positive constant* $C$, *depending only on a and b such that*

$$\left(\frac{\|x^a f(x)\|_\infty}{|\hat{f}(0)|}\right)^{b-1} \left(\frac{\|\xi^b \hat{f}(\xi)\|_\infty}{|f(0)|}\right)^{a-1} \geqslant C,$$

*whenever* $f\colon R \to R$ *is an even continuous and integrable function.*

We observe that the conditions $W(0) = 0 = V(0)$ (implied by (4)) are essential for a non-trivial $L^\infty$ uncertainty principle. Indeed, if $W(0)\, V(0) \neq 0$ and $f(0)\, \hat{f}(0) \neq 0$, then

$$\frac{\| Wf \|_\infty}{|\hat{f}(0)|} \frac{\| V\hat{f} \|_\infty}{|f(0)|} \geqslant W(0)\; V(0).$$

The example below shows that the conditions (4)–(5) are essential for the validity of Theorem 1.

EXAMPLE 1. Let $W(x)=|x|$ and $V(\xi)=\xi^2$. Then there exist no increasing, continuous functions $\Phi$ and $\Psi$ on $[0,\infty)$ with $\Phi(0)=0=\Psi(0)$ and such that (7) is satisfied for all continuous integrable functions $f: R\to R$.

*Proof.* Let $a_n > b_n > 0$ for $n=1, 2, 3, \ldots$ with both $a_n$ and $b_n$ tending to infinity and $b_n a_n^{-1}$ tending to zero. Let $X_n$ and $Y_n$ denote the characteristic functions of the intervals $(-a_n, a_n)$ and $(-b_n, b_n)$ respectively. Define $h_n$ to be the convolution of $X_n$ and $Y_n$. Then

$$h_n(x) := |(-a_n, a_n) \cap (x-b_n, x+b_n)|,$$

where $|\cdot|$ denotes Lebesgue measure. We observe that $h_n$ is an even, non-negative and compactly supported continuous function. Furthermore, $h_n(x)=2b_n$ if $|x| \leqslant a_n - b_n$, $h_x(x)=a_n+b_n-|x|$ if $a_n-b_n \leqslant |x| \leqslant a_n+b_n$, and $h_n(x)=0$ if $|x| \geqslant a_n+b_n$. Moreover

$$\hat{h}(\xi)=4\xi^{-2}\sin(a_n\xi)\sin(b_n\xi).$$

The global maximum of $|x|\, h_n(x)$ is attained when $a_n-b_n, \leqslant |x| < a_n+b_n$. Since $b_n a_n^{-1} \to 0$, the derivative of $|x|\, h_n(x)$ is non-vanishing on the interval $a_n-b_n < |x| < a_n+b_n$, for $n$ sufficiently large. Hence, for these values of $n$, we have

$$\frac{\|xh_n\|_\infty}{|\hat{h}_n(0)|} = \frac{2b_n(a_n-b_n)}{4a_nb_n} \leqslant \frac{1}{2}. \tag{8}$$

On the other hand, we have for all $n$,

$$\frac{\|\xi^2\hat{h}_n\|_\infty}{|h_n(0)|} \leqslant \frac{2}{b_n} \to 0. \tag{9}$$

The inequalities (8) and (9) show that there exist no strictly increasing, continuous functions $\Phi$ and $\Psi$ on $[0,\infty)$ with $\Phi(0)=0=\Psi(0)$ and such that (7) is satisfied for all continuous integrable functions $f: R\to R$.

Q.E.D.

## 3. PROOFS

We shall use a variant of Wiener's criterion for the validity of the Poisson Summation Formula. See for example [3].

LEMMA 1. *Let $W$ be an even function defined on $R$, strictly positive and non-decreasing on $(0, \infty)$, and such that*

$$\int_1^\infty \frac{dt}{W(t)} < \infty.$$

*If $f: R \to C$ is a continuous function such that $Wf \in L^\infty$, then for any nonzero $x$,*

$$\frac{2\pi}{|x|} \sum_{k=-\infty}^{\infty} f(2k\pi/x) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(kx). \tag{10}$$

*Proof of Lemma.* The monotonicity and integrability of $1/W$ on $[1, \infty)$ imply that for each nonzero $x$, $\sum_{k=-\infty}^{\infty} f(t + 2k\pi\,|x|^{-1})$ converges uniformly for $|t| \leqslant \pi/|x|$. Indeed, if $|t| \leqslant \pi/|x|$, then $|t + 2k\pi\,|x|^{-1}| \geqslant (2\,|k| - 1)\,\pi/|x|$. In turn, this implies, for $|k| \geqslant 2$,

$$|f(t + 2k\pi\,|x|^{-1})| \leqslant \frac{\|Wf\|_\infty}{W((2\,|k|-1)\,\pi\,|x|^{-1})} \leqslant \frac{2\pi\,\|Wf\|_\infty}{|x|} \int_{I_k} \frac{ds}{W(s)},$$

where $I_k$ denotes the interval $[(2\,|k| - 3)\,\pi\,|x|^{-1}, (2\,|k| - 1)\,\pi\,|x|^{-1}]$. The integrability of $W^{-1}$ on $(\delta, \infty)$ for any $\delta > 0$ completes the proof of our claim that $\sum_{k=-\infty}^{\infty} f(t + 2k\pi\,|x|^{-1})$ converges uniformly for $|t| \leqslant \pi/|x|$, given any fixed nonzero real number $x$. Hence, the function

$$t \to \sum_{k=-\infty}^{\infty} f(t + 2k\pi\,|x|^{-1})$$

is continuous on $R$ for each $x \neq 0$. Equation (10) follows by applying Fejér's theorem at $t = 0$. Q.E.D.

*Proof of Theorem* 1. Let $f: R \to R$ be a continuous integrable function. For the moment, we assume $f(0)\,\hat{f}(0) < 0$. For nonzero $x$, define

$$\theta_f(x) := \frac{2\pi}{|x|} \sum_{k \in Z^*} f(2\pi k/x) - \hat{f}(0) \tag{11}$$

where $Z^*$ denotes the set of nonzero integers. Using the monotonicity of $W$ and the growth condition (6), we observe that for any $h > 0$,

$$\frac{h}{W(h)} \leqslant \frac{C_0 h}{W(2h)} \leqslant C_0 \int_h^{2h} \frac{dt}{W(t)}.$$

Thus, for any $h>0$, we have

$$\sum_{k=1}^{\infty} \frac{h}{W(kh)} \leqslant \sum_{j=0}^{\infty} \frac{2^j h}{W(2^j h)} \leqslant C_0 \int_h^{\infty} \frac{dt}{W(t)}. \tag{12}$$

The first inequality in (12) is obtained by decomposing the set of positive integers into the disjoint sets $\{2^j+k : 0 \leqslant k < 2^j\}$, $j=0, 1, 2, \ldots$. Multiplying (11) by $\hat{f}(0)$, a real number, we obtain

$$\hat{f}(0)\, \theta_f(x) \leqslant \frac{2\pi}{|x|} \, \|Wf\|_{\infty} \, |\hat{f}(0)| \sum_{k \neq 0} W(2\pi k/x)^{-1} - |\hat{f}(0)|^2.$$

Combining this with (12) yields

$$\hat{f}(0)\, \theta_f(x) \leqslant 2C_0 \, \|Wf\|_{\infty} \, |\hat{f}(0)| \int_{2\pi/|x|}^{\infty} \frac{ds}{W(s)} - |\hat{f}(0)|^2. \tag{13}$$

We observe that conditions (4) and (5) imply

$$\lim_{s \to \infty} \int_{2\pi/s}^{\infty} \frac{dt}{W(t)} = \infty \qquad \text{and} \qquad \lim_{s \to 0^+} \int_{2\pi/s}^{\infty} \frac{dt}{W(t)} = 0$$

respectively. Moreover, the integrability of $W^{-1}$ on $(\delta, \infty)$ for any $\delta > 0$, together with the fact that it takes real positive values there, implies that

$$s \to \int_{2\pi/s}^{\infty} \frac{ds}{W(s)}$$

is a strictly increasing continuous function on $(0, \infty)$. Hence there exists a unique positive $x_0$ satisfying

$$\frac{|\hat{f}(0)|}{\|Wf\|_{\infty}} = 2C_0 \int_{2\pi/x_0}^{\infty} \frac{ds}{W(s)}.$$

With $x = x_0$, the right-hand side of (13) becomes zero. This implies

$$\hat{f}(0)\, \theta_f(x) < 0, \qquad \text{whenever} \quad 0 < |x| < x_0. \tag{14}$$

Next, we combine the Poisson Summation Formula (10) and equation (11) to obtain an alternative expression for $\theta_f$:

$$\theta_f(x) = \lim_{N \to \infty} \sum_{1 \leqslant |k| \leqslant N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(kx) - \frac{2\pi}{|x|} f(0). \tag{15}$$

Applying (12) with $V$ in place of $W$ and $C_1$ in place of $C_0$ yields the following inequality:

$$\left| x \sum_{1 \leqslant |k| \leqslant N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(kx) \right| \leqslant 2C_1 \, \| V\hat{f} \|_\infty \int_{|x|}^{\infty} \frac{dt}{V(t)}.$$

From (15) and the assumption $\hat{f}(0)\, f(0) < 0$, we arrive at the inequality

$$|x|\, \hat{f}(0)\, \theta_f(x) \geqslant 2\pi \, |f(0)\, \hat{f}(0)| - 2C_1 \, |\hat{f}(0)| \cdot \| V\hat{f} \|_\infty \int_{|x|}^{\infty} \frac{dt}{V(t)}.$$

Again, we observe that

$$\lim_{s \to 0^+} \int_s^\infty \frac{dt}{V(t)} = \infty \qquad \text{and} \qquad \lim_{s \to \infty} \int_s^\infty \frac{dt}{V(t)} = 0$$

and $s \to \int_s^\infty \frac{dt}{V(t)}$ is a strictly decreasing continuous function on $(0, \infty)$. Hence, there exist a unique positive $x_1$ such that

$$\frac{\pi \, |f(0)|}{\| V\hat{f} \|_\infty} = C_1 \int_{x_1}^{\infty} \frac{dt}{V(t)}.$$

Moreover, we have

$$|x|\, \hat{f}(0)\, \theta_f(x) > 0, \qquad \text{whenever} \quad |x| > x_1. \tag{16}$$

Comparing (14) and (16), we conclude that the sets $\{x \in R : |x| > x_1\}$ and $\{x \in R : 0 < |x| < x_0\}$ must be disjoint. Hence $x_0 \leqslant x_1$.

Our discussion shows that the mappings $H$ and $G$ defined by

$$H(u) = \left(2C_0 \int_{2\pi u}^{\infty} \frac{ds}{W(s)}\right)^{-1} \qquad \text{and} \qquad G(u) = \left(\frac{C_1}{\pi} \int_u^\infty \frac{ds}{V(s)}\right)^{-1}, \qquad u > 0,$$

are bijections of $(0, \infty)$ with itself. Moreover,

$$H(x_0^{-1}) = \frac{\| Wf \|_\infty}{|\hat{f}(0)|}, \qquad G(x_1) = \frac{\| V\hat{f} \|_\infty}{|f(0)|}.$$

Thus the inequality $x_0 \leqslant x_1$ is equivalent to

$$H^{-1}\left(\frac{\| Wf \|_\infty}{|\hat{f}(0)|}\right) G^{-1}\left(\frac{\| V\hat{f} \|_\infty}{|f(0)|}\right) \geqslant 1 \tag{17}$$

where $f\colon R \to R$ is continuous, even and integrable with $\hat{f}(0)\, f(0) < 0$.

It remains to consider functions $f$ such that $f(0)\,\hat{f}(0)>0$. Fix $\alpha$ such that $1/2<\alpha<1$. Given such an $f$, define $F(x)=\alpha f(x/2)-f(x)$. Then $\hat{F}(\xi)=2\alpha\hat{f}(2\xi)-\hat{f}(\xi)$ and

$$F(0)\,\hat{F}(0)=(\alpha-1)(2\alpha-1)\,f(0)\,\hat{f}(0)<0.$$

Applying (17) to $F$ and observing that

$$\|WF\|_\infty\leqslant(\alpha C_0+1)\,\|Wf\|_\infty,\qquad \|V\hat{F}\|_\infty\leqslant(2\alpha+1)\,\|V\hat{f}\|_\infty$$

and that $H^{-1}$ and $G^{-1}$ are increasing functions, we obtain

$$H^{-1}\left(A_\alpha\frac{\|Wf\|_\infty}{|\hat{f}(0)|}\right)G^{-1}\left(B_\alpha\frac{\|V\hat{f}\|_\infty}{|f(0)|}\right)\geqslant 1 \tag{18}$$

where $A_\alpha=(\alpha C_0+1)(2\alpha-1)^{-1}$ and $B_\alpha=(2\alpha+1)(1-\alpha)^{-1}$.

Finally, we define $\Phi(t)=H^{-1}(A_\alpha t)$, $\Psi(t)=G^{-1}(B_\alpha t)$. Since $A_\alpha$ and $B_\alpha$ are both greater than one, (17) shows that (18) also holds whenever $f(0)\,\hat{f}(0)<0$. Trivially, (18) holds when $f(0)\,\hat{f}(0)=0$. Q.E.D.

## REFERENCES

1. J. Benedetto, Frame decompositions, sampling, and uncertainty principle inequalities, *in* "Wavelets: Mathematics and Applications" (J. Benedetto and M. Frazier, Eds.), pp. 247–303, CRC Press, Boca Raton, 1994.
2. G. B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey, *J. Fourier Anal. Appl.* **3** (1997), 207–238.
3. J. P. Kahane and P. G. Lemarié-Rieusett, Remarques sur la formule sommatoire de Poisson, *Studia Math.* **109** (1994), 303–316.
4. Y. Katznelson, "An Introduction to Harmonic Analysis," Wiley, New York, 1968, reprinted by Dover, New York, 1976.